# Final Exam Solutions

## Instructions

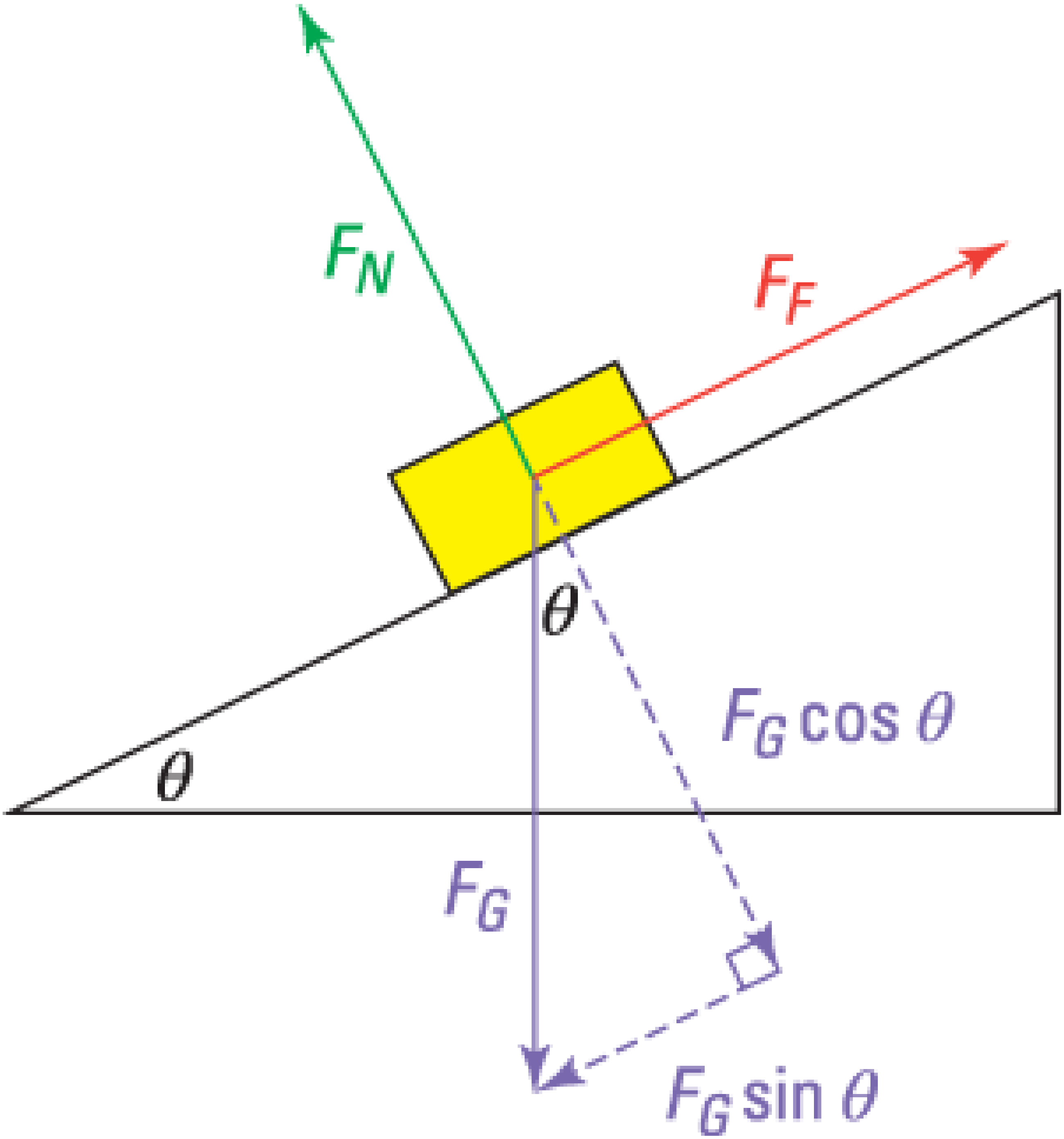
- Write your name on this test booklet.
- You are allowed a writing utensil, a calculator, and the PHY 201 formula sheet.
- You will have the full class period to complete this exam.
- Show all of your work.
- Good luck!

## Problem 1

(15 points) A 15 kg box starts from rest and slides down a 22 meter long ramp at an angle of $37^\circ$ with respect to the horizontal. How fast is the box moving at the bottom of the ramp? The coefficient of friction between the box and the ramp is 0.33.

**Solve this problem using forces, Newton's laws, and kinematics.**

We have seen this setup a number of times now. We can draw a free-body diagram for the setup as follows:




I will call the direction along the ramp $\hat{x}$ and the direction perpendicular to the ramp $\hat{y}$. Summing the forces in the y-direction yields:

$$F_N\,(+\hat{y}) + F_G cos\theta\,(-\hat{y}) = 0$$

$$F_N - F_G cos\theta = 0$$

$$F_N = mgcos\theta$$

Summing the forces along the x-direction yields:

$$F_G\, sin\theta\, (+\hat{x}) + F_f\, (-\hat{x}) = ma\, (+\hat{x})$$

$$mg\, sin\theta\, (+\hat{x}) + \mu F_N\, (-\hat{x}) = ma\, (+\hat{x})$$

$$mg\, sin\theta\, (+\hat{x}) + \mu mg\, cos\theta\, (-\hat{x}) = ma\, (+\hat{x})$$

$$mg\, sin\theta - \mu mg\, cos\theta = ma$$

$$g\, sin\theta - \mu g\, cos\theta = a$$

We know from our kinematics equations that $v_f^2 = v_o^2 + 2a\Delta x$. Thus:

$$v_f^2 = v_o^2 + 2a\Delta x$$

$$v_f^2 = 2a\Delta x$$

$$v_f^2 = 2(g\, sin\theta - \mu g\, cos\theta)\Delta x$$

$$v_f = \sqrt{2g\, sin\theta\, \Delta x - 2\mu g\, cos\theta\, \Delta x}$$

Plugging in values yields:

$$v_f \approx 12.1\, m/s$$

## Problem 2

(15 points) A 15 kg box starts from rest and slides down a 22 meter long ramp at an angle of $37^\circ$ with respect to the horizontal. How fast is the box moving at the bottom of the ramp? The coefficient of friction between the box and the ramp is 0.33.

**Solve this problem using energy, forces, and work.**

Let's start by writing out the full expression for conservation of energy:

$$K_i + U_i + W_{nc} = K_f + U_f$$

At the top of the ramp, we have potential, but no kinetic energy. At the bottom of the ramp, we have kinetic, but no potential energy (assuming we set $U_G = 0$ at the bottom of the ramp). Thus:

$$U_i + W_{nc} = K_f$$

$$mgh + W_{nc} = \frac{1}{2}mv^2$$

The non-conservative work comes from the work done by friction over the 22 meter long ramp.

$$mgh + \left[\vec{F} \cdot \Delta\vec{x}\right] = \frac{1}{2}mv^2$$

$$mgh + [F\ \Delta x\ cos(\alpha)] = \frac{1}{2}mv^2$$

The friction force is in the opposite direction as the displacement. Thus:

$$mgh + [F\ \Delta x\ cos(180^\circ)] = \frac{1}{2}mv^2$$

$$mgh - F\Delta x = \frac{1}{2}mv^2$$

From the previous problem, we know that the force of friction is $F = \mu N$.

$$mgh - \mu N \Delta x = \frac{1}{2}mv^2$$

$$mgh - \mu mg\ cos\theta\ \Delta x = \frac{1}{2}mv^2$$

$$gh - \mu g\ cos\theta\ \Delta x = \frac{v^2}{2}$$

We can get the height $h$ using $sin(\theta) = h/\Delta x$

$$g\ sin\theta\ \Delta x - \mu g\ cos\theta\ \Delta x = \frac{v^2}{2}$$

$$v = \sqrt{2g\ sin\theta\ \Delta x - 2\mu g\ cos\theta\ \Delta x}$$

Plugging in values yields:

$$v \approx 12.1\ m/s$$

## Problem 3

(5 points) I am standing still and see a car drive past me with a velocity of $20\hat{x}$ m/s. Simultaneously, I see you ride past me on a bike with a velocity of $-5\hat{x}$ m/s. What is the speed of the car from your frame of reference?

We will let $\vec{u}'$ be the velocity of the car in your frame of reference, $\vec{u}$ be the velocity of the car in my frame of reference, and $\vec{v}$ be the speed of your reference frame with respect to mine. Thus:

$$\vec{u} = \vec{u}' + \vec{v}$$

$$20\hat{x}\ m/s = \vec{u}' - 5\hat{x}\ m/s$$

$$\vec{u}' = 25\hat{x}\ m/s$$

Problem 4

(10 points) A thin rod with length L = 1.0 m and mass M = 15 kg is hinged on one side so that it can rotate freely. A wad of putty with a mass of 0.55 kg is thrown so that it hits the exact centre of the rod with a speed of 20 m/s and sticks. What is the angular velocity of the rod just after the putty sticks to it? You can treat the wad of putty as a point mass.

We can use the law of conservation of angular momentum to solve this problem. Initially, we have only the angular momentum from the putty, whereas after the collision we have the angular momentum from the rod and putty together. The putty hits the rod at its center - $L/2$:

$$L_i = L_f$$

$$m_{putty}v_{putty}r = I_{total}\omega$$

$$m_{putty}v_{putty}\left(\frac{L}{2}\right) = \left[\frac{1}{3}m_{rod}L_{rod}^2 + m_{putty}\left(\frac{L}{2}\right)^2\right]\omega$$

$$m_{putty}v_{putty}\left(\frac{L}{2}\right) = \left[\frac{1}{3}m_{rod}L_{rod}^2 + m_{putty}\left(\frac{L}{2}\right)^2\right]\omega$$

$$5.5\ kg \cdot m^2/s = \left[5.1375\ kg \cdot m^2\right]\omega$$

$$\omega \approx 1.1\ rad/s$$

Problem 5

(5 points) The pressure at the surface of a lake is one atmosphere (101,325 Pa). What is the pressure at a depth of 10 meters? The density of water is 997 $kg/m^3$.

$$P_{10m} = P_{surface} + \rho g h$$

$$P_{10m} = 101325\ Pa + (997\ kg/m^3)\left(9.8\ m/s^2\right)(10\ m)$$

$$P_{10m} = 199,031\ Pa$$

## Problem 6

(5 points) A garden hose of radius 1.0 cm carries water at 2.0 m/s. The nozzle at the end of the hose has a radius of 0.3 cm. What is the speed of the water as it exits the hose?

We know that the fluid in the hose is incompressible, so we have equal volumes of water entering and exiting the hose per unit time:

$$A_1 v_1 = A_2 v_2$$

$$\pi(0.01m)^2(2.0m/s) = \pi(0.003m)^2 v_2$$

$$v_2 \approx 22.2m/s$$

## Problem 7

(5 points) What does it mean when I say that an object undergoes an elastic deformation? How is that different from a plastic deformation?

A plastic deformation is one where you expend energy to change the shape of some material, and it stays changed. Conversely, an elastic deformation is one where the material goes back to its original form after you deformed it. Thus, mechanical energy is lost whenever an object undergoes a plastic deformation.

## Problem 8

(5 points) A mass ($m = 5\ kg$) is attached to a spring ($k = 50\ N/m$) and oscillates back and forth with an amplitude of 0.10 m. What is the maximum speed of this mass-spring oscillator and where does it occur along the path of the mass?

We can solve for the maximum speed using the law conservation of energy. At the amplitude of the motion, the energy is all in the form of spring potential energy (no kinetic energy, no speed). At the very center of the oscillation when the spring is unstretched, we get the maximum kinetic energy. Thus:

$$U_{spring} = K$$

$$\frac{1}{2}kA^2 = \frac{1}{2}mv_{max}^2$$

$$v_{max} = \sqrt{\frac{kA^2}{m}}$$

$$v_{max} = A\sqrt{\frac{k}{m}}$$

$$v_{max} = (0.10\ m)\sqrt{\frac{50\ N/m}{5\ kg}} \approx 0.32m/s$$

## Problem 9

(5 points) The period of the oscillation for a mass-spring oscillator is 1.5 seconds, and its amplitude is 0.10 m. What is the magnitude of the maximum acceleration of the oscillator?

The equation of motion for the oscillator is:

$$x(t) = Asin(\omega t + \phi)$$

Taking the first and second derivative gives us velocity and acceleration:

$$v(t) = \omega Acos(\omega t + \phi)$$

$$a(t) = -\omega^2 Asin(\omega t + \phi)$$

Which means the maximum acceleration is equal to:

$$a_{max} = \omega^2 A$$

The value for $\omega$ can be found from the period:

$$\omega = \frac{2\pi}{T}$$

Thus:

$$a_{max} = \frac{4\pi^2 A}{T^2} \approx 1.75m/s^2$$

## Problem 10

(5 points) Why is it that the astronauts in orbit are said to be weightless, even though the force of gravity on them is only around 10% less than it is on Earth's surface?

We say that the astronauts are weightless because there is no external contact force acting on them while they are in orbit. Only the force of gravity is acting on the astronaut. Thus, the astronaut is essentially in free-fall as they orbit the Earth.

## Problem 11

(5 points) The gravitational potential energy around a massive object is given by $U_G = -GMm/R$ (this comes from Newton's Law of Gravitation). However, we also learned that the gravitational potential energy close to the surface of the Earth is given by $U_G = mgh$. How do you get one expression from the other? You don't have to do any math - a conceptual explanation is fine.

We started with the gravitational potential energy around a massive body and broke the radius $R$ up into two parts - the radius of the planet Earth plus the height $h$ above the Earth. We then performed a Taylor series expansion around $h/R$ and made the approximation that $h/R$ was very small. This simplified down to $U_g = mgh$.

## Problem 12

(5 points) What is the difference between the mechanical energy and the total energy of a system? Why do we care about this distinction?

The total energy of a system includes all forms of energy present in the system (e.g. kinetic, chemical, nuclear, electromagnetic, etc). Mechanical energy is equal to sum of the kinetic energy and potential energy. We care about mechanical energy because it can be directly converted into useful mechanical work. Thus, mechanical energy is also defined as the ability to do work.

## Problem 13

(10 points) A particle moves at a constant speed of 0.45 m/s in a circle of radius 0.80 meters centered at the origin. Write an expression for the x-component of the particle's position as a function of time. You can assume that the particle is located at the point (0.00 m, 0.80 m) at time t = 0 seconds.

Uniform circular motion can be thought of as simple harmonic motion along the x-axis plus simple harmonic motion along the y-axis. We know that the particle starts at $x = 0$ at time $t = 0$. Thus, along the x-axis:

$$x(t) = Asin(\omega t)$$

The amplitude is given as $A = 0.80\,m$. The value for $\omega$ can be calculated from $\omega = v/r = 0.5625\,rad/s$. Thus:

$$x(t) = (0.80\,m)sin\,[(0.5625\,rad/s)t]$$

### Problem 14

(5 points) What does it mean for a mass-spring oscillator to be critically damped?

In chapter 14, we looked at linear damping in a qualitative fashion, and defined three types of damping - underdamped, overdamped, and critically damped. Underdamping means the oscillator will still oscillate, but its amplitude will decrease over time. Overdamped means that there is so much damping that the oscillator does not oscillate, but instead moves slowly towards the equilibrium position. Critical damping means that there is no oscillation, and the oscillator gets back to equilibrium as quickly as possible.